Instanton theory of Burgers shocks and intermittency

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(Received 15 December 2008; published 29 April 2009)

A Lagrangian approach to Burgers turbulence is carried out along the lines of the field theoretical Martin-Siggia-Rose formalism of stochastic hydrodynamics. We derive, from an analysis based on the hypothesis of unbroken Galilean invariance, the asymptotic form of the probability distribution function of negative velocity differences. The origin of Burgers intermittency is found to rely on the dynamical coupling between shocks, identified to instantons, and noncoherent background fluctuations, which—then—cannot be discarded in a consistent statistical description of the flow.

DOI: 10.1103/PhysRevE.79.046324 PACS number(s): 47.27.eb, 47.27.ef

I. INTRODUCTION

The long established Burgers model of compressible one-dimensional flow [1] provides an interesting testing ground for the performance of numerical and analytical strategies in turbulence research. Despite its own peculiar phenomenology, as evidenced by the complete failure of approximations based on the "K41" scaling theory [2], there are important conceptual analogies between the Burgers model and the usual three-dimensional turbulence (see Ref. [3] for a comprehensive review).

We note, in passing, that the Burgers model is more than just a "mathematical toy;" in its multidimensional version the Burgers equation plays an important role in several realistic problems such as nonlinear acoustics [4], cosmology [5,6], critical interface growth [7], and traffic flow dynamics [8].

A great deal of attention has been focused on the problem of non-Gaussian fluctuations observed in the high Reynolds number regime of Burgers dynamics—the intermittency phenomenon, for short. As it is verified through numerical simulations [9,10], velocity differences

$$z = u(x+\zeta,t) - u(x-\zeta,t) \quad (1)$$

are found to be very intermittent at small scales (ζ much lesser than the integral scale L). The probability distribution function (pdf) of the right tail of z, which decays faster than Gaussian, has been analytically obtained in a number of different ways [10–13]. The viscous left tail (far left tail) also decays faster than Gaussian, as derived by Balkovsky *et al.* [14]. In the inviscid limit, however, the left tail is related to Burgers shocks and is found to have a power-law profile $\rho(z) \sim 1/|z|^{\alpha}$, with no sharp consensus on the value taken by the exponent α.

A Fokker-Planck approach to the computation of velocity-difference pdfs, with closure given by an operator product expansion treatment of the dissipative anomaly was put forward by Polyakov [12]. This method provides a fine description of the pdf's right tail and yields a power-law form for the left tail with $5/2 \le \alpha \le 3$ [15]. Extensive numerical simulations performed by Gotoh and Kraichnan [10] indicate that $\alpha=3$. At variance with such findings, an analytical study based on the velocity field profiles in space-time neighborhoods of shocks—the so-called preshock events—gives $\alpha=7/2$ [13,16], a result confirmed by alternative Lagrangian simulations of the Burgers equation [17,18].

In an attempt to conceal these apparently contradictory conclusions, Boldyrev *et al.* [19] suggested that the left tail exponent is not universal, departing from $\alpha=3$ if flow realizations fail to satisfy a strong form of Galilean invariance, which holds—by definition—if usual Galilean invariance is observed in the bulk, regardless the boundary conditions at infinity. In rephrased form, the whole point of Ref. [19] is that finite-size effects which break strong Galilean invariance would lock larger fluctuations of shock jumps and negative velocity derivatives, reducing intermittency. In this paper, we find support to the conjecture that the left tail exponent is $\alpha=3$ when the strong form of Galilean invariance is fulfilled.

This work is organized as follows. In Sec. II, we introduce Burgers intermittency as a phenomenon related, in the inviscid limit, to shock amplitude fluctuations. The great convenience of a Lagrangian description of the flow is then pointed out. In Sec. III, we discuss the Martin-Siggia-Rose (MSR) formulation of stochastic hydrodynamics [20,21] within the Lagrangian perspective. In Sec. IV, Burgers shocks will be given as instantons [11,22], and background fluctuations around them will be taken into account in the computation of the asymptotic behavior of the pdf of negative velocity differences. In Sec. V, we summarize and discuss our results. In Appendixes A and B we provide technical details on some of the material discussed in Sec. IV.

II. PHENOMENOLOGICAL CONSIDERATIONS

The Burgers model describes the dynamics of a one-dimensional velocity field $u=u(x,t)$ ruled by the evolution equation

$$\partial_t u + u\partial_x u = \nu \partial_x^2 u + f, \quad (2)$$

where ν is the kinematical viscosity and $f=f(x,t)$ is the external force which sustains the flow and introduces the integral length scale L. There is no pressure term in the above equation and no imposition of incompressibility as well.

Let $u_0(x) \equiv u(x,0)$ be the velocity field at initial time supposed to be a C^1 function defined on $-\infty < x < \infty$. The Cauchy problem is exactly solvable for the Burgers equation

[23,24]. In the forceless case, the velocity field is, at time $t>0$,

$$u(x,t) = -2\nu\partial_x \times \ln\left\{\int_{-\infty}^{\infty} dy \exp\left[-\frac{(x-y)^2}{4\nu t} - \frac{1}{2\nu}\int_0^y dx' u_0(x')\right]\right\}. \quad (3)$$

As it is well known, Eq. (3) leads, in the vanishing viscosity limit, to discontinuous shocks (i.e., the velocity field becomes piecewise C^1), which can be interpreted as "sinks" of fluid particles. External forcing does not spoil the process of shock creation, even though it can affect the statistics of shock amplitudes.

Stable and unstable regions of the flow are distinguished, essentially, by the sign of the spatial velocity derivative. Neglecting higher-order corrections, consider the expansion $u(x,t)=\sigma_0(t)+\sigma_1(t)x$ in the neighborhood of an arbitrary point. Equation (2) gives, for $f(x,t)=0$,

$$\dot{\sigma}_0 + \sigma_0\sigma_1 = 0,$$

$$\dot{\sigma}_1 + \sigma_1^2 = 0, \quad (4)$$

leading to

$$\sigma_0(t) = \sigma_0(0)\exp\left[-\int_0^t dt' \sigma_1(t')\right], \quad (5)$$

where

$$\sigma_1(t) = \frac{\sigma_1(0)}{1+\sigma_1(0)t}. \quad (6)$$

Therefore, if $\sigma_1(0)=\partial_x u|_{t=0}$ is positive, we expect from Eq. (6) that $\partial_x u$ will decay. On the other hand, if $\sigma_1(0)$ is negative then $|\partial_x u|$ will increase in time, implying flow instability. This is the mechanism for the generation of large negative velocity derivatives in Burgers turbulence, which in a time scale of order $1/|\partial_x u|$ are transformed into long-lived shocks.

We are interested to study the statistics of negative velocity differences z for $\nu \to 0$ and $\zeta/L \ll 1$. Under these circumstances, shock jumps provide the main contribution to the strong fluctuations of z. A time series of z would exhibit intermittent negative spikes associated with the passage of shocks separated by much weaker signals due to smooth velocity configurations.

Suppose, now that fluctuations of z are alternatively measured from the subtraction of velocity fields defined at points $x(t)+\zeta$ and $x(t)-\zeta$, where $x(t)$ is the position of a fluid element that moves with the flow. We expect to have, in such a Lagrangian framework, the same asymptotic power-law form for the left tail pdf of z. The central point in this correspondence is that fluid particles typically spend a finite fraction of their times at shock discontinuities. Once a fluid particle is dragged into a shock discontinuity, it remains there until the shock collapses or it is absorbed by another one.

Negative spikes in the Eulerian time series of z are replaced, in the Lagrangian reference frame, by smooth fluctuations of shock amplitudes, which last for much longer times. In order to compute statistical properties of the Eulerian negative spikes, one would have to find out how shocks match to each other in solutions of the Burgers equation. Within the Lagrangian framework, on the other hand, it suffices to describe fluctuations around isolated shocks—a much simpler task that points out, for our purposes, the advantage of Lagrangian methods over Eulerian ones.

III. PATH-INTEGRAL FRAMEWORK

In the stochastic hydrodynamics approach to Burgers turbulence, the external forcing in Eq. (2) is taken to be a large scale Gaussian random field, with zero mean and correlator

$$\langle f(x,t)f(x',t')\rangle = D(|x-x'|)\delta(t-t'), \quad (7)$$

where we take

$$D(|x-x'|) = D_0 \exp(-|x-x'|^2/L^2). \quad (8)$$

The conditional probability density functional to have velocity configuration $u_0(x)$ at time $t=0$ if $u_{-T}(x)$ is the velocity at time $t=-T$ can be written as the path integral [20,21]

$$Z = \mathcal{N}\int D\hat{u}Du \exp(iS), \quad (9)$$

where $\mathcal{N}$ is an unimportant normalization factor (which will be usually supressed from notation) and

$$S = S[\hat{u},u] \equiv \int_{-T}^0 dt \int dx \hat{u}[\partial_t u + u\partial_x u - \nu\partial_x^2 u] + \frac{i}{2}\int_{-T}^0 dt \int dx dx' \hat{u}(x,t)\hat{u}(x',t)D(|x-x'|) \quad (10)$$

is the so-called MSR action. Expressions (9) and (10) are subject to the boundary conditions

$$u_0(x) = u(x,0),$$

$$u_{-T}(x) = u(x,-T). \quad (11)$$

In order to pave the way for a Lagrangian description of the flow, let us consider a general reference frame R' which moves with velocity $\phi(t)$ relative to the original (inertial) "laboratory" frame R. The position and velocity in R' are

$$x' = x - \int_{-T}^t dt' \phi(t'), \quad (12)$$

$$u_\phi(x',t) = u(x,t) - \phi(t). \quad (13)$$

In the moving frame, the velocity at the origin $(x'=0)$ is defined as

$$u_\phi(t) \equiv u_\phi(0,t) = u\left[\int_{-T}^t dt' \phi(t'),t\right] - \phi(t). \quad (14)$$

For a given field $u=u(x,t)$, there is a unique time-dependent function $\phi(t)$ which solves $u_\phi(0,t)=0$. It is clear that $\phi(t)$ is in this case the velocity of a locally comoving reference frame—that is how Lagrangian coordinates come into play. We introduce, correspondingly, the Faddeev-Popov determinant [25] $\Delta[u(0,t)]$ by means of

$$\Delta^{-1}[u(0,t)] \equiv \int D\phi \delta[u_\phi(t)]. \tag{15}$$

Note that $\Delta[u(0,t)]$ is invariant under the generalized Galilean transformations given by Eqs. (12) and (13). In fact,

$$\Delta^{-1}[u_{\phi_0}(t)] \equiv \int D\phi \delta[u_{\phi+\phi_0}(t)] = \int D\phi \delta[u_\phi(t)] = \Delta^{-1}[u(0,t)]. \tag{16}$$

Relation (15) yields

$$\Delta[u(0,t)] \int D\phi \delta[u_\phi(t)] = 1. \tag{17}$$

Inserting Eq. (17) into the integrand of Eq. (9) and exchanging the order of integrations, we get

$$Z = \int D\phi \int D\hat{u} Du \Delta[u(0,t)] \delta[u_\phi(t)] \exp(iS). \tag{18}$$

Generalized Galilean transformations can be used to replace the Dirac's delta functional in Eq. (18) by $\delta[u_{\phi=0}(t)] = \delta[u(0,t)]$. To accomplish that, we first substitute, in the MSR action of Eq. (18), the integration fields $u(x,t)$ and $\hat{u}(x,t)$ by Galilean transformed ones through

$$u(x,t) = u_\phi(x',t) + \phi(t), \tag{19}$$

$$\hat{u}(x,t) = \hat{u}_\phi(x',t). \tag{20}$$

We find

$$S = S_\phi + \int_{-T}^{0} dt \int dx \hat{u}_\phi \frac{d\phi}{dt}, \tag{21}$$

where

$$S \equiv S[\hat{u}(x,t), u(x,t)], \tag{22}$$

$$S_\phi \equiv S[\hat{u}_\phi(x,t), u_\phi(x,t)]. \tag{23}$$

The additional term on the right-hand side (RHS) of Eq. (21) takes account of the noninertial force due to the acceleration $\dot{\phi}$ of the reference frame R'.

The Jacobian associated with the above transformations is unity, as can be verified from the matrix elements

$$\hat{O}(x_1,x_2|t_1,t_2) \equiv \frac{\delta \hat{u}(x_1,t_1)}{\delta \hat{u}_\phi(x_2,t_2)} = \frac{\delta u(x_1,t_1)}{\delta u_\phi(x_2,t_2)} = \delta\left[x_1 - x_2 + \int_{-T}^{t_1} dt' \phi(t')\right] \delta(t_1 - t_2). \tag{24}$$

The operator which has the matrix elements given by (24) can be written in any reasonable functional space of space-time-dependent functions as

$$\hat{O} = \exp\left[\int_{-T}^{t} dt' \phi(t') \frac{\partial}{\partial_x}\right]. \tag{25}$$

The eigenstates of $\hat{O}$ are the momentum wave functions $\exp(ipx)$. Using a parity-preserving discretization of the Fourier space, the Jacobian turns out to be

$$\det[\hat{O}] = \prod_p \exp\left[ip \int_{-T}^{t} dt' \phi(t')\right] = 1. \tag{26}$$

The Faddeev-Popov determinant $\Delta[u(0,t)]$ is also unity. In fact, consider the velocity field which has been "gauge fixed," with the help of a generalized Galilean transformation, to $u(0,t)=0$. We have, then, to substitute the functional Taylor expansion of $u_\phi(t)$ up to first order in $\phi(t)$ in Eq. (15). Defining $g(t)=\partial_x u(x,t)|_{x=0}$, we get

$$u_\phi(t) = g(t) \int_{-T}^{t} dt' \phi(t') - \phi(t) + O[\phi^2(t)], \tag{27}$$

so that

$$\Delta^{-1}[u(0,t)] \equiv \int D\phi \delta\left[g(t) \int_{-T}^{t} dt' \phi(t') - \phi(t)\right] = |\det[\delta(t-t') - \Theta(t-t') g(t)]|^{-1}. \tag{28}$$

Using, now, the identity

$$\det[X] = \exp[\mathrm{Tr}(\ln X)], \tag{29}$$

we find, with $A(t,t') \equiv \Theta(t-t') g(t)$,

$$\Delta[u(0,t)] = \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \mathrm{Tr}(A^n)\right] = 1, \tag{30}$$

since

$$\mathrm{Tr}(A^n) = \int dt_1 dt_2 \dots dt_n g(t_1) g(t_2) \dots g(t_n) \times \Theta(t_1 - t_2) \Theta(t_2 - t_3) \dots \Theta(t_n - t_1) = 0. \tag{31}$$

Above, we have used $\Theta(0)=0$, which is the right prescription for the Heaviside function when the underlying stochastic differential equation is defined in terms of an Itô discretized time evolution [26]. Collecting all of the above results, we rewrite Eq. (18) as

$$Z = \int D\phi \int D\hat{u}_\phi Du_\phi \delta[u_\phi(t)] \exp\left(iS_\phi + i \int_{-T}^{0} dt \frac{d\phi}{dt} \int dx \hat{u}_\phi\right). \tag{32}$$

However, since $\hat{u}_\phi(x,t)$ and $u_\phi(x,t)$ are integration fields, Eq. (32) becomes

$$Z = \int D\phi \int D\hat{u}Du\,\delta[u(0,t)]\exp\left(iS + i\int_{-T}^{0} dt \frac{d\phi}{dt}\int dx\hat{u}\right), \tag{33}$$

or, equivalently, integrating over $\phi(t)$,

$$Z = \int D\hat{u}Du\,\delta\left[\int dx\hat{u}(x,t)\right]\delta[u(0,t)]\exp(iS). \tag{34}$$

In view of Eq. (33), we will assume in all our subsequent considerations that $u(0,t)=0$. In other words, we have moved for good to the locally comoving reference frame. As fluid elements happen to stick (and spend a finite fraction of their times) in shock discontinuities, the latter will be frequently hosted at the origin of the locally comoving reference frame.

We stress, at this point, that the hypothesis of strong Galilean invariance [19] is a fundamental ingredient here, since no role is given to the velocity boundary conditions at infinity, in the Lagrangian formulation put forward in Eq. (33).

IV. SHOCKS AND INTERMITTENCY

The response functional (34) can be decomposed as

$$Z = Z_s + Z_{\bar{s}}, \tag{35}$$

where Z_s and $Z_{\bar{s}}$ refer, respectively, to the cases where shocks and smooth field configurations are found at $x=0$, $t=0$. Recalling the discussion of Sec. II, it is clear that the statistical properties of large negative velocity differences are all encoded in Z_s.

According to the probabilistic interpretation of Z, we note that Z_s is not normalized to unit. Instead, Z_s is normalized to the "intermittency factor" γ, where $0<\gamma<1$ is the fraction of time a shock is found at the origin of the locally comoving reference frame.

Let $u_s(x)$ be a shock configuration, with velocity discontinuity at $x=0$, at its instant of creation [27]. Assuming that the shock creation is uniformly distributed in time, one may write, for the probability density functional to get configuration $u_0(x)$ at time $t=0$ (see Appendix A),

$$Z_s = \gamma\int_0^\infty \frac{d\eta}{\eta}\int_0^\eta dT \int Du_s(x)P[u_s(x)]W[\eta,T;u_0(x),u_s(x)], \tag{36}$$

where $Du_s(x)P[u_s(x)]$ is the probability measure for the creation of the shock $u_s(x)$ conditioned to be in the sample space of all shock creation events and $W[\eta,T;u_0(x),u_s(x)]$ is a weighting functional.

Equation (36) is formally rigorous, but it is of hard practical implementation, due to the difficulty in getting information on the functionals $P[u_s(x)]$ and $W[\eta,T;u_0(x),u_s(x)]$. Phenomenological arguments, however, can be helpful in order to replace Eq. (36) by more tractable expressions.

Shocks are expected to have (i) mean interdistances of the order of the integral scale L and (ii) lifetimes of the order of $T=L/U$, where U is an estimate of the shock velocity jump. The prototypical Burgers shock is the stationary configuration

$$u_s(x;U) = -U\tanh\left(\frac{U}{2\nu}x\right). \tag{37}$$

Even though Eq. (37) is a solution of the forceless Burgers equation, it can be used as a local approximation to general viscous shocks around the position of velocity discontinuity. Suppose that at time $-T$ a configuration similar to Eq. (37) is created and let $g(U)$ be the probability density that it has amplitude U. Due to property (ii) above, this shock is not going to be observed at time $t=0$ if $T \gtrsim L/U$. The contribution to Eq. (36) provided by shocks with the local profile (37) is, then, estimated as

$$Z_s = \gamma\int_0^\infty dUg(U)\frac{U}{L}\int_0^{L/U} dT\mathcal{N}\int D\hat{u}Du\;\delta \left[\int dx\hat{u}(x,t)\right]\delta[u(0,t)]\exp(iS), \tag{38}$$

where the velocity field $u(x,t)$ satisfies the boundary condition

$$u(x,-T) = u_s(x;U). \tag{39}$$

Expression (38) is, in fact, straigthforwardly obtained from Eq. (A6), taking the substitutions $T[u_s]\to L/U$ and $\int Du_s(x)P[u_s(x)]\to\int_0^\infty dUg(U)$.

We will work with the phenomenologically simplified result (38) in place of Eq. (36). In doing so, we conjecture that the asymptotic statistical properties of negative velocity differences are not affected by more detailed choices of shock parametrization.

An interesting way to address the computation of Eq. (38) is to perform the corresponding path integration over an appropriate subset of the functional space, which would consist of dominating configurations. That is precisely the purpose of the saddle-point method applied in Refs. [11,22] to the MSR turbulence context as a way to cope with the intermittency phenomenon.

Saddle-point configurations dubbed instantons are associated with stationary values of the action. Taking functional derivatives of the MSR action (10) with respect to the integration fields, we get

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = -i\int dx'\hat{u}(x',t)D(|x-x'|) \tag{40}$$

and

$$\partial_t\hat{u} + u\partial_x\hat{u} + \nu\partial_x^2\hat{u} = 0. \tag{41}$$

When solving Eqs. (40) and (41), we have to take into account the constraints [28]

$$u(0,t) = 0, \tag{42}$$

$$\int dx\hat{u}(x,t) = 0. \tag{43}$$

Instanton solutions of Eqs. (40) and (41), which hold for $-T<t<0$ and satisfy Eqs. (39), (42), and (43) can be readily obtained,

$$u(x,t) = u_s(x;U),$$

$$\hat{u}(x,t) = 0. \quad (44)$$

It is worth mentioning that the solution for $u(x,t)$ in Eq. (44) identifies Burgers shocks to instantons. Also, it is not difficult to find that the MSR action vanishes when evaluated for the fields given in Eq. (44). As it is the standard procedure in the saddle-point method, we expand the MSR action in a functional Taylor series around the instantons, retaining only quadratic fluctuations. We replace, as a result, Eq. (10) by

$$S^\star = \int_{-T}^{0} dt \int dx \hat{u}[\partial_t u + \partial_x(u_s u) - \nu \partial_x^2 u] + \frac{i}{2}\int_{-T}^{0} dt \int dx dx' \hat{u}(x,t)\hat{u}(x',t) D(|x-x'|), \quad (45)$$

where the velocity boundary condition becomes now $u(x,-T)=0$.

In order to compute the pdf of negative velocity differences, we introduce the characteristic function

$$Z_s(\lambda) = \gamma \int_0^\infty dU g(U) \frac{U}{L}\int_0^{L/U} dT \mathcal{N} \int D\hat{u} Du \ \delta \times \left[\int dx \hat{u}(x,t)\right] \delta[u(0,t)] \exp(iS^\star - i\lambda z), \quad (46)$$

where z is the velocity-difference evaluated at $t=0$,

$$z = -2U + u(\zeta,0) - u(-\zeta,0). \quad (47)$$

The characteristic function $Z_s(\lambda)$ can be exactly computed, in principle, since it is given in Eq. (46) by a quadratic field theory. To evaluate $Z_s(\lambda)$, the saddle-point method can be applied once again, this time in an exact way. The further saddle-points equations for $u(x,t)$ and $\hat{u}(x,t)$ are

$$\partial_t u + \partial_x(u u_s) - \nu \partial_x^2 u = -i \int dx' \hat{u}(x',t) D(|x-x'|), \quad (48)$$

$$\partial_t \hat{u} + u_s \partial_x \hat{u} + \nu \partial_x^2 \hat{u} = \lambda[\delta(x+\zeta) - \delta(x-\zeta)]\delta(t), \quad (49)$$

supplemented by Eqs. (42) and (43).

Observe that the viscosity term has the "wrong" sign in Eq. (49). To avoid the unbounded growing of $\hat{u}(x,t)$ for $t>0$, we impose, as prescribed in Refs. [11,22], the boundary condition $\hat{u}(x,0^+)=0$. Integrating Eq. (49) over the time interval $[-\epsilon,\epsilon]$, with $\epsilon \to 0$, we get the "final condition"

$$\hat{u}(x,0^-) = \lambda[\delta(x-\zeta) - \delta(x+\zeta)]. \quad (50)$$

Furthermore, we have the exact saddle-point result (see Appendix B)

$$\bar{S}^\star - \lambda \bar{z} = 2\lambda U + \frac{i}{2}\int_{-T}^{0} dt \int dx dx' \hat{u}(x,t)\hat{u}(x',t) D(|x-x'|). \quad (51)$$

It is interesting to note, due to Eq. (51), that we do not have to worry in finding the specific solution for $u(x,t)$. Equation (49) is solved, in the vanishing viscosity limit, by

$$\hat{u}(x,t) = \lambda\{\delta[x - x(t)] - \delta[x + x(t)]\}, \quad (52)$$

where $x(t)=\zeta - Ut$. Substituting Eq. (52) into Eq. (51) and taking $\zeta/L \ll 1$, we find

$$\bar{S}^\star - \lambda \bar{z} = 2\lambda U + i\frac{D_0 L}{2U}\lambda^2 \int_{-2UT/L}^{0} dt[1 - e^{-t^2}]. \quad (53)$$

We get, from Eq. (46)

$$Z_s(\lambda) = \gamma \int_0^\infty dU g(U) \int_0^1 d\eta \exp\left(2i\lambda U - \frac{D_0 L c(\eta)}{2U}\lambda^2\right), \quad (54)$$

where

$$c(\eta) = \int_{-2\eta}^{0} dt[1 - e^{-t^2}]. \quad (55)$$

The negative velocity-difference pdf is computed from the Fourier transform of the characteristic function, as

$$\rho(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \exp(i\lambda z) Z_s(\lambda) = \gamma \int_0^\infty dU g(U) \int_0^1 d\eta \sqrt{\frac{U}{2\pi c(\eta) D_0 L}} \times \exp\left[-\frac{U(z+2U)^2}{2c(\eta)D_0 L}\right]. \quad (56)$$

Expression (56) gives, for large negative z, the asymptotic pdf

$$\rho(z) = \frac{a}{|z|^3} + \dots, \quad (57)$$

where the dots refer to subleading contributions, and

$$a = \gamma L D_0 g(0) \int_0^1 d\eta c(\eta) \simeq 0.36 \gamma L D_0 g(0). \quad (58)$$

The expression for the coefficient (58) is a testable prediction of the present theory. Alternative force-force correlation functions can be used to recompute Eq. (58) and compare it with the value to be found in further numerical simulations. It is clear that in the Eulerian framework, the intermittency factor γ has to be replaced by

$$\gamma' = 2n\zeta, \quad (59)$$

where n is the number of shocks per unit length.

V. CONCLUSIONS

We have obtained, with the help of instanton techniques, the asymptotic form of the pdf of large negative velocity differences in Burgers turbulence. The Lagrangian picture of the Burgers flow was adapted to the MSR field theoretical framework, a procedure which proved to be an important technical improvement over the Eulerian description. Lagrangian methods are, as a rule, welcome in the study of small-scale intermittency, since they cope in a natural way with the sweeping produced by large scale motions. In the case of Burgers flow, sweeping produces shock advection, making it difficult to find out the statistical properties of velocity-difference fluctuations.

The introduction of Lagrangian coordinates was carried out under the hypothesis of strong Galilean invariance. We have found that the left tail pdf has the asymptotic form $\rho(z)=a/|z|^3$, which agrees with the conjecture put forward in Ref. [19], that this is so when strong Galilean invariance holds. We have obtained an explicit expression for the critical amplitude a, which motivates the study of further numerical simulations of Burgers turbulence.

Arbitrary shocks of the Burgers forceless model are identified to instantons and taken, in the path-integral formulation of the response functional, as the dominant configurations for the determination of the velocity-difference fluctuations. We have bypassed the detailed classification of all of these Burgers shocks at their creation events, by noting that relevant parameters of newborn shocks are their velocity jump, U, and extension, assumed to scale with the integral length L. Shocks are expected to have lifetimes of the order of L/U. In view of the role of the dimensional parameters U and L, we take the stationary Burgers shock (44) more as an illustration than as an essential ingredient in the formalism.

We emphasize that the instanton distribution $g(U)$ is not able to yield the left tail pdf of velocity differences on its own. The point is that the fluctuating background couples with the shocks and by the usual instability mechanism discussed in Sec. II, large negative velocity differences are generated in the flow. One may wonder if this process of intermittency generation is analogously found in the interaction between the background and coherent structures in Navier-Stokes turbulence.

An interesting problem where the instanton approach discussed here could be applied concerns the "artificial multiscaling" induced by stochastic forcing. As the authors of Ref. [29] show with the help of numerical simulations, stochastic forcing with a $1/k$ spectrum leads to logarithmic corrections in the structure functions which can be confused with anomalous exponents. It is possible that the combination of inviscid and viscous instanton methods may throw some light on this issue from the theoretical point of view.

ACKNOWLEDGMENTS

This work has been partially supported by CNPq and FAPERJ.

APPENDIX A: DERIVATION OF Eq. (36)

Let us focus our attention on a flow which evolves under a particular realization of the stochastic force $f(x,t)$. We also suppose that a random ensemble of initial configurations is given in the remote past $(t\rightarrow-\infty)$, so that at any instant of time the possible velocity configurations yield a statistically stationary ensemble.

We consider, now, the creation at time $-T$ of a shock $u_s(x)$ localized at the origin of the locally comoving reference frame, which survives until time $t=0$. Let $T[u_s,f]$ be the maximum value of T. Since the probability for the creation of the shock $u_s(x)$ in a time interval dT is also proportional to dT, we may write

$$Z_s = \gamma \int Du_s(x)P[u_s(x)] \times \left\langle \frac{1}{T[u_s,f]}\int_0^{T[u_s,f]} dTP(u_0,u_s;0,-T[u_s,f]) \right\rangle_f , \tag{A1}$$

where $P(u_0,u_s;0,-T[u_s,f])$ is the probability distribution associated with the transition from the shock configuration $u_s(x)$, which was created at time $-T$, to the final configuration (at time $t=0$) $u_0(x)$. For the sake of clarity, we note that

$$P(u_0,u_s;0,-T[u_s,f]) = \delta\{u_0(x) - L([u_s,f];x,T)\}, \tag{A2}$$

where $L([u_s,f];x,T)$ is the velocity configuration which evolves from the shock $u_s(x)$ after the time interval T.

Equation (A1) can be rewritten as Eq. (36). In fact,

$$\begin{aligned} Z_s &= \gamma \int Du_s(x)P[u_s(x)] \\ &\quad \times \int_0^\infty \frac{d\eta}{\eta}\int_0^\eta dT\langle \delta(\eta - T[u_s,f])P[u_0,u_s;0,-T]\rangle_f \\ &= \gamma \int_0^\infty \frac{d\eta}{\eta}\int_0^\eta dT \int Du_s(x)P[u_s(x)]W[\eta,T;u_0(x),u_s(x)], \end{aligned} \tag{A3}$$

where

$$W[\eta,T;u_0(x),u_s(x)] = \langle \delta(\eta - T[u_s,f])P[u_0,u_s;0,-T]\rangle_f. \tag{A4}$$

If, as an approximation, $T[u_s,f]$ does not depend on $f(x,t)$, then

$$W[\eta,T;u_0(x),u_s(x)] = \delta(\eta - T[u_s])\langle P[u_0,u_s;0,-T]\rangle_f \tag{A5}$$

and, therefore,

$$Z_s = \gamma \int Du_s(x)P[u_s(x)]\frac{1}{T[u_s]}\int_0^{T[u_s]} dT\langle P[u_0,u_s;0,-T]\rangle_f. \tag{A6}$$

Above, the averaged probability $\langle P[u_0,u_s;0,-T]\rangle_f$ can be given as the MSR path-integral expression (34).

APPENDIX B: DERIVATION OF Eq. (51)

There is some subtlety in the saddle-point evaluation of characteristic functionals such as Eq. (46). Since we are considering in Eq. (46) the evolution up to time $t=0$, one could object that the $\hat{u}(x,0^+)=0$ boundary condition sounds too loose. Actually, in order to apply the saddle-point method to Eq. (46), the time evolution is extended to $t \to \infty$. Saddle-point solutions are, then, such that $u(x,t\to\infty)=\hat{u}(x,t\to\infty)=0$. Note that the time extension does not change the value of $Z_s(\lambda)$, once velocity configurations are integrated out at $t\to\infty$ in the path integral (46).

Taking these remarks into account, we multiply both sides of Eq. (49) by $u(x,t)$ and integrate them over space and time. We find

$$\int dx \int_{-T}^{\infty} dt \hat{u}[\partial_t u + u_s \partial_x u - \nu \partial_x^2 u] = \lambda[u(\zeta,0) - u(-\zeta,0)], \tag{B1}$$

where we have used the boundary conditions $u(\pm\infty,t)=0$, $\hat{u}(x,t>0)=0$, and $u(x,-T)=0$. Equation (51) follows immediately from Eqs. (45), (47), and (B1).

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